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Derivation of low-temperature expansions for Ising model X. The four-dimensional simple hypercubic lattice

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Abstract. The derivation of high-field expansions for the four-dimensional simple hypercubic lattice is described briefly. The high-field polynomials L_n are given up to L_{15} together with the complete partial generating functions (codes) up to F_7 which determine the corresponding sub-lattice polynomials. Expansions are given for the zero-field free energy and initial susceptibility in powers of the high-temperature counting variable $v = \tanh K$ up to v^{17}, and combined with the codes these determine the susceptibility and all its field derivatives up to v^{17}.

1. Introduction

In this paper we apply the theory and techniques of previous papers (Sykes *et al* 1965, 1973a, b, c, d, e, 1975a, b, c, hereafter referred to as I–IX respectively) to the four-dimensional simple hypercubic lattice. We derive the complete partial generating functions (codes) F_0–F_7 and these provide sufficient information to determine all the high-field sub-lattice polynomials (defined in II, equation 1.6) up to 15th order in the field variables μ and ν. We give the high-field polynomials up to $L_{15}(\mu)$ explicitly. We also extend the low-temperature (or energy) grouping and extract the expansions of the free energy in zero field, the spontaneous magnetisation and the initial susceptibility up to u^{35}. The complete codes determine implicitly the high-temperature expansion for the free energy in a magnetic field in powers of the standard high-temperature counting variable $v = \tanh K$ up to v^{15}. By extending the high-temperature expansions for the zero-field free energy and initial susceptibility up to v^{17} we provide sufficient information to determine the free energy in a magnetic field up to v^{17}.

It is our main object to communicate the new data; the length of the calculations makes it impractical to report them in detail. A more general introduction and a justification of our interest in the problem is given in an accompanying paper (Gaunt *et al* 1979).

2. Derivation of complete codes for the four-dimensional simple hypercubic lattice

We treat the four-dimensional simple hypercubic lattice as a generalisation of the simple quadratic lattice whose code system we have studied in I, II, III and IV and of the simple cubic lattice whose code system we have studied in I, II, V and VI.

We follow Fisher and Gaunt (1964) and study the four-dimensional simple hypercubic lattice whose sites are given by four *integer* coordinates (p_1, p_2, p_3, p_4) where the p_i take all possible combinations of positive and negative values. The lattice sites can be divided into disjoint sets: an A sub-lattice formed by the points (a_1, a_2, a_3, a_4), Σa_i even, and a B sub-lattice formed by the points (b_1, b_2, b_3, b_4), Σb_i odd. The method of partial generating functions (described in I and II) exploits the fact that any nearest neighbour of a site on the A sub-lattice lies on the B sub-lattice and conversely.

Any excited spin on the A sub-lattice intersects with its eight neighbouring spins on the B sub-lattice. The solid figure formed by this first neighbour shell is thought of as a shadow; to derive the nth order partial generating function the possible overlappings of the shadows of n excited spins have to be classified and counted.

If we take the first excited spin as origin, the shadows cast by any of the 24 nearest points on the A sub-lattice

$$(a_1, a_2, a_3, a_4) \qquad |a_i| = 1 \text{ or } 0, \qquad \sum |a_i| = 2 \tag{1.1}$$

will have two points in common with the shadow cast by the origin. Overlappings of this type can be thought of as corresponding to the first-neighbour bonds of the four-dimensional face-centred cubic defined by the sub-lattice A. (The bond length corresponding to (1.1) is $\sqrt{2}a$, if a denotes the intersite distance for the original hypercubic lattice.)

The shadows corresponding to the eight points

$$(a_1, a_2, a_3, a_4) \qquad |a_i| = 2 \text{ or } 0, \qquad \sum |a_i| = 2 \tag{1.2}$$

will have one point in common with the shadow cast by the origin. Overlappings of this type can be thought of as corresponding to the bonds of four disjoint four-dimensional hypercubic lattices which correspond to second-neighbour interactions of the face-centred cubic lattice A.

There are no other possible overlappings of the shadows of a pair of spins and the *shadow lattice* is therefore a four-dimensional face-centred cubic lattice with second-neighbour bonds *as defined above*. It is to be noticed that our definition of second-neighbour bonds is a restricted one, dictated by the shadow classification problem; the four-dimensional face-centred cubic has other second-neighbour bonds forming a set of disjoint four-dimensional body-centred cubic lattices. Explicitly, the shadows corresponding to the 16 points

$$(a_1, a_2, a_3, a_4) \qquad |a_i| = 1 \tag{1.3}$$

have no points in common with the origin shadow but they are at the same cartesian distance ($\sqrt{4}a = 2a$) as the set (1.2).

The code system generated by the overlappings of increasing numbers of shadows is essentially similar to that of the three-dimensional simple cubic (V, § 3) but exhibits certain extra features. It is configurationally more complex; for example, strong embeddings of a square can occur on the shadow lattice. This property is a little puzzling at first sight since such embeddings are clearly not possible in the three-dimensional face-centred cubic with second-neighbour bonds; it is in fact only possible because of our disregard of the second-neighbour bonds corresponding to (1.3). It is also found that in four dimensions the strong embeddings of a tetrahedron correspond to two distinct codes and thus give rise to a parity problem of the kind already noticed for the honeycomb and diamond codes. (III, § 2; V, § 2).

We illustrate in table 1 the possible star embeddings in the shadow lattice of up to four points in two, three and four dimensions and their corresponding codes. The total numbers of graphical codes (III, § 2) that occur up to seventh order are summarised in table 2. It will be seen that the increase in passing from three to four dimensions is only fractional; however, the counting problem is made more difficult by the large increase in the actual number of possible embeddings of each code due to the higher coordination number.

Table 1. Codes corresponding to star embeddings in the shadow lattice in two, three and four dimensions for the simple cubic system. Second-neighbour bonds are primed.

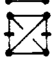
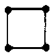
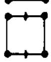
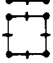
Star embedding	Code $d=2$	Code $d=3$	Code $d=4$
	(4, 4)	(6, 6)	(8, 8)
	(6, 4, 2)	(10, 8, 2)	(14, 12, 2)
	(7, 6, 1)	(11, 10, 1)	(15, 14, 1)
	None	(13, 9, 3, 1)	(19,15, 3, 1)
	(8, 5, 2, 1)	(14, 11, 2, 1)	(20, 17, 2, 1)
	None	(16, 12, 0, 4)	(23, 16, 6, 0, 1) (24, 20, 0, 4)
	None	(16, 10, 5, 0, 1)	(24, 18, 5, 0, 1)
	(9, 4, 4, 0, 1)	(17, 12, 4, 0, 1)	(25, 20, 4, 0, 1)
	None	(16, 10, 4, 2)	(24, 8, 4, 2)
	None	(17, 12, 3, 2)	(25, 20, 3, 2)
 	(10, 6, 2, 2)	(18, 14, 2, 2)	(26, 22, 2, 2)
	None	None	(24, 16, 8)
	None	(18, 12, 6)	(26, 20, 6)
	(12, 8, 4)	(20, 16, 4)	(28, 24, 4)

Table 2. The total numbers of graphical codes in the first seven complete codes for two, three and four dimensions.

	$d=2$	$d=3$	$d=4$
F_1	1	1	1
F_2	3	3	3
F_3	6	7	7
F_4	12	17	19
F_5	21	36	45
F_6	39	77	102
F_7	63	153	222

To obtain the first seven complete codes we have proceeded along the general lines used previously for the diamond, simple cubic and body-centred cubic lattices (V, §§ 2, 3). Connected embeddings have been counted by computer; as before we have exploited the principle of complete code balance to find the contribution of some separated graphs. We give the codes F_0–F_7 and high-field polynomials L_1–L_{15} derived from them in the appendix. The polynomials, which determine the critical exponent δ characterising the critical isotherm, will be analysed in a forthcoming paper.

3. Derivation of complete codes for general dimension

By a straightforward generalisation of the investigation of the preceding section the general d-dimensional simple hypercubic lattice $(p_1, p_2, \ldots, p_d)$ can be defined and the corresponding code system derived. The codes in table 1 may be expressed easily as linear functions of d; the counting of the embeddings in d dimensions, although straightforward, becomes quite difficult as the order of the code increases. We quote the result for the first three complete codes; these determine the first seven high-field polynomials for all dimensions.

$$
\begin{aligned}
F_1 &= \binom{d}{0}(2d, 2d) \\
F_2 &= 2\binom{d}{2}(4d-2, 4d-4, 2) + \binom{d}{1}(4d-1, 4d-2, 1) - \left[2\binom{d}{2} + \binom{d}{1} + \tfrac{1}{2}\binom{d}{0}\right](4d, 4d) \\
F_3 &= 8\binom{d}{3}(6d-5, 6d-9, 3, 1) + 4\binom{d}{2}(6d-4, 6d-7, 2, 1) \\
&\quad + \left[2\binom{d}{2} + 24\binom{d}{3} + 48\binom{d}{4}\right](6d-4, 6d-8, 4) \\
&\quad + \left[8\binom{d}{2} + 24\binom{d}{3}\right](6d-3, 6d-6, 3) \\
&\quad + \left[\binom{d}{1} - 20\binom{d}{2} - 96\binom{d}{3} - 96\binom{d}{4}\right](6d-2, 6d-4, 2) \\
&\quad - \left[4\binom{d}{1} + 20\binom{d}{2} + 24\binom{d}{3}\right](6d-1, 6d-2, 1) \\
&\quad + \left[\tfrac{1}{3}\binom{d}{0} + 3\binom{d}{1} + 26\binom{d}{2} + 64\binom{d}{3} + 48\binom{d}{4}\right](6d, 6d).
\end{aligned}
\tag{3.1}
$$

4. Low-temperature grouping

As we have shown in earlier papers (IV, VI) the information contained in complete codes is not always well suited to the derivation of expansions in zero field ($\mu = 1$); however, this becomes progressively less so as the dimension, and therefore the coordination number, increases. The complete codes F_0–F_7 suffice to develop expansions up to u^{31}. To supplement these we give the leading terms of those high-field polynomials that contribute to higher powers of u through u^{35}:

$$\begin{aligned} L_{16} &= u^{32}+0u^{33}+0u^{34}+960u^{35}+\dots \\ L_{17} &= 64u^{35}+\dots. \end{aligned} \tag{4.1}$$

The leading term in L_{16} corresponds to 16 spins arranged as a hypercube. Combining the above terms with those derived from the codes we obtain the expansions for the reduced configurational free energy, spontaneous magnetisation and ferromagnetic susceptibility (defined in VI, § 4):

$$\begin{aligned} \ln \Lambda = {} & u^4+4u^7-4\tfrac{1}{2}u^8+28u^{10}-64u^{11}+42\tfrac{1}{3}u^{12} \\ & +228u^{13}-834u^{14}+1116u^{15}+1477\tfrac{3}{4}u^{16}-10\,404u^{17} \\ & +21\,460u^{18}-1956u^{19}-119\,035\tfrac{4}{5}u^{20}+358\,697\tfrac{1}{3}u^{21} \\ & -344\,316u^{22}-1\,132\,588u^{23}+5\,421\,143\tfrac{1}{2}u^{24} \\ & -9\,187\,444u^{25}-5\,820\,150u^{26}+73\,867\,260u^{27} \\ & -184\,905\,115\tfrac{6}{7}u^{28}+95\,069\,292u^{29}+869\,893\,575\tfrac{1}{3}u^{30} \\ & -3\,217\,644\,924u^{31}+4\,228\,381\,183\tfrac{7}{8}u^{32}+7\,584\,693\,262\tfrac{2}{3}u^{33} \\ & -49\,870\,463\,026u^{34}+103\,034\,060\,812\tfrac{4}{5}u^{35}+\dots \end{aligned} \tag{4.2}$$

$$\begin{aligned} I(u) = {} & 1-2u^4-16u^7+18u^8-168u^{10}+384u^{11}-266u^{12}-1824u^{13} \\ & +6672u^{14}-9216u^{15}-15\,522u^{16}+74920u^{17}-219\,240u^{18} \\ & -6640u^{19}+1\,433\,114u^{20}-4\,364\,368u^{21}+4\,015\,104u^{22} \\ & +16\,249\,856u^{23}-76\,650\,222u^{24}+129\,304\,000u^{25}+107\,955\,904u^{26} \\ & -1\,194\,988\,848u^{27}+2\,988\,132\,104u^{28}-1\,295\,881\,792u^{29} \\ & -16\,000\,351\,200u^{30}+58\,541\,360\,096u^{31}-74\,808\,889\,446u^{32} \\ & -161\,492\,842\,096u^{33}+1\,010\,237\,004\,872u^{34} \\ & -2\,065\,384\,405\,984u^{35}+\dots \end{aligned} \tag{4.3}$$

$$\begin{aligned} \tfrac{1}{4}\chi_0(u) = {} & u^4+16u^7-18u^8+252u^{10}-576u^{11}+423u^{12}+3648u^{13} \\ & -13\,344u^{14}+19\,152u^{15}+40\,294u^{16}-259\,440u^{17}+562\,020u^{18} \\ & +94\,264u^{19}-4\,307\,021u^{20}+13\,303\,104u^{21}-5632\,992u^{22} \\ & -57\,973\,856u^{23}+271\,183\,914u^{24}-455\,166\,560u^{25}-482\,496\,656u^{26} \\ & +4\,831\,425\,864u^{27}-12\,082\,047\,182u^{28}+4\,185\,256\,128u^{29} \\ & +73\,401\,528\,384u^{30}-266\,390\,485\,184u^{31}+330\,255\,999\,226u^{32} \\ & +851\,455\,636\,352u^{33}-5\,115\,138\,468\,772u^{34} \\ & +10\,351\,608\,527\,640u^{35}+\dots. \end{aligned} \tag{4.4}$$

These expansions are being analysed and the results will be published shortly.

5. High-temperature zero-field expansions

To derive the high-temperature zero-field expansion for the free energy we have studied its star cluster expansion. The general theory of this method is given by Sykes *et al* (1974) and Sykes and Hunter (1974); the compilation of the star lattice constant list required presents special problems which have been described in general by Sykes *et al* (1966). For the special configurational problems in four dimensions we have drawn extensively on studies undertaken during recent investigations of the percolation problem in higher dimensions by Gaunt *et al* (1976) and Gaunt and Ruskin (1978). In terms of the standard high-temperature counting variable $v = \tanh K$ we find for the configurational free energy

$$\ln \Lambda(v) = \ln 2 - 4 \ln(1+v) + 6v^4 + 76v^6 + 1371v^8 + 30\,152v^{10} + 758\,496v^{12} + 20\,956\,508v^{14} + 619\,695\,597\tfrac{1}{2}v^{16} + \mathrm{O}(v^{18}). \qquad (5.1)$$

To derive the high-temperature initial susceptibility we have used the star cluster expansion for its inverse. The method derives from the early studies of Yvon (1945) and has been developed by Domb and Hiley (1962) and more recently by Domb (private communications) and McKenzie (1976). The configurational problem and the range of the star list required is closely related to that for the free energy. We find

$$\chi_0(v) = 1 + 8v + 56v^2 + 392v^3 + 2696v^4 + 18\,536v^5 + 126\,536v^6 + 863\,720v^7 + 5\,873\,768v^8 + 39\,942\,184v^9 + 271\,009\,112v^{10} + 1\,838\,725\,896v^{11} + 12\,457\,092\,504v^{12} + 84\,392\,312\,392v^{13} + 571\,140\,732\,808v^{14} + 3\,865\,210\,690\,888v^{15} + 26\,138\,072\,412\,040v^{16} + 176\,752\,645\,540\,264v^{17} + \ldots \qquad (5.2)$$

The expansion (5.2) is in agreement with Fisher and Gaunt (1964) up to v^{11}. The expansions (5.1) and (5.2) are in agreement with those derived independently from the codes up to v^{15}. The coefficient of v^{16} in (5.1), together with the codes F_0–F_7, determines the coefficient of v^{16} in (5.2) and these are in agreement. Finally, the coefficients of v^{17} in (5.1) and (5.2) are not checked by the codes, nor one by the other, but combined with the codes they determine the free energy in a field (and therefore all its field derivatives) up to v^{17}. The zero-field expansion of the fourth-field derivative, $\chi_0^{(2)}$, is given in the accompanying paper (Gaunt *et al* 1979) where χ_0 and $\chi_0^{(2)}$ are both analysed.

The theoretical justification for the above statements rests on the high–low manipulation which connects the two classes of expansion. Some details of this manipulation are given by Domb (1949, 1974) (see also I, § 2 and IX, § 2 and references therein) and it is hoped to publish further details subsequently.

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Appendix. Complete codes F_n and high-field polynomials L_n for the four-dimensional simple hypercubic lattice

$F_1 = 1(8, 8)$

$F_2 = 12(14, 12, 2) + 4(15, 14, 1) - 16\frac{1}{2}(16, 16)$

$F_3 = 32(19, 15, 3, 1) + 156(20, 16, 4) + 24(20, 17, 2, 1) + 144(21, 18, 3)$
$\quad -596(22, 20, 2) - 232(23, 22, 1) + 472\frac{1}{3}(24, 24)$

$F_4 = 16(23, 16, 6, 0, 1) + 12(24, 16, 8) + 144(24, 18, 4, 2) + 48(24, 18, 5, 0, 1)$
$\quad +8(24, 20, 0, 4) + 864(25, 19, 5, 1) + 192(25, 20, 3, 2)$
$\quad +6(25, 20, 4, 0, 1) + 2276(26, 20, 6) + 1296(26, 21, 4, 1)$
$\quad +48(26, 22, 2, 2) + 3744(27, 22, 5) - 1704(27, 23, 3, 1)$
$\quad -15\,000(28, 24, 4) - 1704(28, 25, 2, 1) - 15\,084(29, 26, 3)$
$\quad +29\,244(30, 28, 2) + 12\,708(31, 30, 1) - 17\,114\frac{1}{4}(32, 32)$

$F_5 = 32(27, 17, 9, 0, 0, 1) + 192(28, 19, 7, 1, 1) + 24(28, 19, 8, 0, 0, 1)$
$\quad +48(28, 20, 4, 4) + 64(28, 21, 3, 3, 1) + 192(29, 19, 9, 1)$
$\quad +384(29, 20, 8, 0, 1) + 672(29, 21, 5, 3) + 29, 21, 6, 1, 1)$
$\quad +96(29, 23, 2, 3, 1) + 288(30, 20, 10) + 192(30, 21, 8, 1)$
$\quad +5232(30, 22, 6, 2) + 1600(30, 22, 7, 0, 1) + 1440(30, 23, 4, 3)$
$\quad +576(30, 23, 5, 1, 1) + 336(30, 24, 2, 4) + 384(31, 22, 9)$
$\quad +20\,064(31, 23, 7, 1) + 11\,520(31, 24, 5, 2) - 160(31, 24, 6, 0, 1)$
$\quad +672(31, 25, 3, 3) + 48(31, 25, 4, 1, 1) + 160(31, 26, 1, 4)$
$\quad +35\,064(32, 24, 8) + 45\,432(32, 25, 6, 1) - 4\,560(32, 26, 4, 2)$
$\quad -3\,720(32, 26, 5, 0, 1) + 72(32, 27, 2, 3) - 634(32, 28, 0, 4)$
$\quad +87\,376(33, 26, 7) - 80\,904(33, 27, 5, 1) - 15\,264(33, 28, 3, 2)$
$\quad -504(33, 28, 4, 0, 1) - 332\,040(34, 28, 6) - 155\,536(34, 29, 4, 1)$
$\quad -4320(34, 30, 2, 2) - 608\,784(35, 30, 5) + 73\,384(35, 31, 3, 1)$
$\quad +1\,077\,176(36, 32, 4) + 103\,608(36, 33, 2, 1) + 1\,209\,264(37, 34, 3)$
$\quad -1\,473\,420(38, 36, 2) - 704\,128(39, 38, 1) + 707\,518\frac{1}{5}(40, 40)$

$F_6 = 24(30, 16, 13, 0, 0, 0, 1) + 4(31, 18, 12, 0, 0, 0, 1) + 48(32, 20, 10, 0, 2)$
$\quad +480(32, 20, 10, 1, 0, 1) + 192(32, 21, 7, 3, 1) + 96(32, 22, 6, 2, 2)$
$\quad +224(32, 22, 6, 3, 0, 1) + 8(32, 24, 0, 8) + 48(32, 24, 2, 4, 2)$

$$
\begin{aligned}
&+96(33, 20, 12, 0, 1)+672(33, 21, 11, 0, 0, 1)+1536(33, 22, 8, 2, 1)\\
&+384(33, 22, 9, 0, 2)+1056(33, 22, 9, 1, 0, 1)+384(33, 23, 5, 5)\\
&+576(33, 23, 6, 3, 1)+384(33, 24, 4, 4, 1)+576(33, 24, 5, 2, 2)\\
&+96(33, 24, 5, 3, 0, 1)+24(34, 20, 14)+2064(34, 22, 10, 2)\\
&+384(34, 22, 11, 0, 1)+384(34, 23, 8, 3)+5952(34, 23, 9, 1, 1)\\
&+1072(34, 23, 10, 0, 0, 1)+5024(34, 24, 6, 4)+8832(34, 24, 7, 2, 1)\\
&+888(34, 24, 8, 0, 2)+384(34, 24, 8, 1, 0, 1)+576(34, 25, 4, 5)\\
&+2592(34, 25, 5, 3, 1)+960(34, 26, 3, 4, 1)+384(34, 26, 4, 2, 2)\\
&+24(34, 28, 0, 4, 2)+5760(35, 23, 11, 1)+3680(35, 24, 9, 2)\\
&+8880(35, 24, 10, 0, 1)+30\,816(35, 25, 7, 3)+34\,752(35, 25, 8, 1, 1)\\
&-2296(35, 25, 9, 0, 0, 1)+10\,176(35, 26, 5, 4)+11\,040(35, 26, 6, 2, 1)\\
&+240(35, 26, 7, 0, 2)+1056(35, 27, 3, 5)+5280(35, 27, 4, 3, 1)\\
&+384(35, 28, 2, 4, 1)+6640(36, 24, 12)+14\,208(36, 25, 10, 1)\\
&+159\,312(36, 26, 8, 2)+43\,776(36, 26, 9, 0, 1)+94\,656(36, 27, 6, 3)\\
&+25\,728(36, 27, 7, 1, 1)-2136(36, 27, 8, 0, 0, 1)+13\,200(36, 28, 4, 4)\\
&+3264(36, 28, 5, 2, 1)+12(36, 28, 6, 0, 2)+864(36, 29, 2, 5)\\
&-3456(36, 29, 3, 3, 1)+16\,440(37, 26, 11)+432\,672(37, 27, 9, 1)\\
&+470\,352(37, 28, 7, 2)-6176(37, 28, 8, 0, 1)+21\,504(37, 29, 5, 3)\\
&-65\,664(37, 29, 6, 1, 1)+12\,448(37, 30, 3, 4)+168(37, 30, 4, 2, 1)\\
&-9000(37, 31, 2, 3, 1)+566\,844(38, 28, 10)+1\,332\,576(38, 29, 8, 1)\\
&-323\,652(38, 30, 6, 2)-215\,360(38, 30, 7, 0, 1)-116\,304(38, 31, 4, 3)\\
&-55\,728(38, 31, 5, 1, 1)-41\,496(38, 32, 2, 4)+1\,909\,368(39, 30, 9)\\
&-2\,617\,272(39, 31, 7, 1)-1\,496\,820(39, 32, 5, 2)-64\,604(39, 32, 6, 0, 1)\\
&-67\,656(39, 33, 3, 3)-4\,944(39, 33, 4, 1, 1)-20\,696(39, 34, 1, 4)\\
&-6\,867\,348(40, 32, 8)-8\,088\,288(40, 33, 6, 1)-211\,164(40, 34, 4, 2)\\
&-7848(40, 35, 2, 3)+233\,568(40, 34, 5, 0, 1)+40\,994(40, 36, 0, 4)\\
&-19\,894\,380(41, 34, 7)+4\,600\,648(41, 35, 5, 1)+955\,200(41, 36, 3, 2)\\
&+33\,708(41, 36, 4, 0, 1)+30\,788\,432(42, 36, 6)+13\,399\,224(42, 37, 4, 1)\\
&+302\,496(42, 38, 2, 2)+66\,495\,292(43, 38, 5)-2\,645\,112(43, 39, 3, 1)\\
&-68\,899\,936(44, 40, 4)-6\,114\,312(44, 41, 2, 1)-88\,175\,662\tfrac{2}{3}(45, 42, 3)\\
&+75\,982\,068(46, 44, 2)+39\,740\,976(47, 46, 1)-31\,827\,819\tfrac{1}{2}(48, 48)
\end{aligned}
$$

$$
\begin{aligned}
F_7 = {}&8(32, 13, 18, 0, 0, 0, 0, 1)+384(35, 19, 14, 1, 0, 0, 1)+288(35, 21, 10, 3, 0, 0, 1)\\
&+336(36, 20, 15, 0, 0, 0, 1)+192(36, 21, 13, 0, 1, 1)
\end{aligned}
$$

$+720(36, 21, 13, 1, 0, 0, 1)+288(36, 22, 10, 2, 2)$

$+576(36, 22, 10, 3, 0, 1)+576(36, 23, 9, 2, 1, 1)$

$+32(36, 23, 9, 3, 0, 0, 1)+64(36, 25, 3, 7, 1)$

$+192(36, 25, 5, 3, 3)+384(36, 25, 5, 4, 1, 1)$

$+16(36, 28, 0, 4, 4)+192(37, 21, 15, 0, 0, 1)$

$+432(37, 22, 14, 0, 0, 0, 1)+768(37, 23, 11, 1, 2)$

$+4128(37, 23, 11, 2, 0, 1)+1152(37, 23, 12, 0, 1, 1)$

$+96(37, 23, 12, 1, 0, 0, 1)+2688(37, 24, 8, 4, 1)$

$+1344(37, 24, 9, 2, 2)+768(37, 24, 9, 3, 0, 1)$

$+192(37, 25, 6, 5, 1)+1440(37, 25, 7, 3, 2)$

$+2304(37, 25, 7, 4, 0, 1)+576(37, 25, 8, 1, 3)$

$+2400(37, 25, 8, 2, 1, 1)+192(37, 26, 4, 6, 1)$

$+384(37, 26, 5, 4, 2)+192(37, 27, 2, 7, 1)$

$+192(37, 27, 3, 5, 2)+192(37, 27, 3, 6, 0, 1)$

$+576(37, 27, 4, 3, 3)+96(37, 27, 4, 4, 1, 1)$

$+96(38, 22, 14, 2)+2304(38, 23, 13, 1, 1)$

$+288(38, 23, 14, 0, 0, 1)+576(38, 24, 10, 4)$

$+1536(38, 24, 11, 2, 1)+1376(38, 24, 12, 0, 2)$

$+13\,152(38, 24, 12, 1, 0, 1)-2136(38, 24, 13, 0, 0, 0, 1)$

$+16\,704(38, 25, 9, 3, 1)+8352(38, 25, 10, 1, 2)$

$+15\,120(38, 25, 10, 2, 0, 1)+1440(38, 25, 11, 0, 1, 1)$

$+2992(38, 26, 6, 6)+11\,136(38, 26, 7, 4, 1)$

$+6480(38, 26, 8, 2, 2)+7776(38, 26, 8, 3, 0, 1)$

$+384(38, 27, 4, 7)+2496(38, 27, 5, 5, 1)+8096(38, 27, 6, 3, 2)$

$+3936(38, 27, 6, 4, 0, 1)+1536(38, 27, 7, 1, 3)$

$+672(38, 27, 7, 2, 1, 1)+288(38, 28, 2, 8)$

$+192(38, 28, 3, 6, 1)+2112(38, 28, 4, 4, 2)$

$+192(38, 29, 2, 5, 2)+48(38, 29, 2, 6, 0, 1)$

$+192(38, 29, 3, 3, 3)+96(38, 29, 3, 4, 1, 1)$

$+768(39, 23, 15, 1)+2688(39, 24, 14, 0, 1)$

$+17\,952(39, 25, 11, 3)+11\,104(39, 25, 12, 1, 1)$

$+14\,304(39, 25, 13, 0, 0, 1)+6144(39, 26, 9, 4)$

$+63\,744(39, 26, 10, 2, 1)+13\,248(39, 26, 11, 0, 2)$

$+42\,720(39, 26, 11, 1, 0, 1) - 376(39, 26, 12, 0, 0, 0, 1)$
$+35\,808(39, 27, 7, 5) + 96\,480(39, 27, 8, 3, 1)$
$+24\,096(39, 27, 9, 1, 2) + 13\,632(39, 27, 9, 2, 0, 1)$
$+192(39, 27, 10, 0, 1, 1) + 6048(39, 28, 5, 6)$
$+34\,048(39, 28, 6, 4, 1) + 25\,248(39, 28, 7, 2, 2)$
$+8288(39, 28, 7, 3, 0, 1) + 7200(39, 29, 4, 5, 1)$
$+11\,328(39, 29, 5, 3, 2) + 960(39, 29, 5, 4, 0, 1)$
$+576(39, 29, 6, 1, 3) + 192(39, 30, 1, 8)$
$+192(39, 30, 2, 6, 1) + 1344(39, 30, 3, 4, 2)$
$+288(39, 31, 2, 3, 3) + 696(40, 24, 16)$
$+576(40, 25, 14, 1) + 76\,416(40, 26, 12, 2)$
$+15\,744(40, 26, 13, 0, 1) + 60\,000(40, 27, 10, 3)$
$+181\,152(40, 27, 11, 1, 1) + 34\,272(40, 27, 12, 0, 0, 1)$
$+237\,120(40, 28, 8, 4) + 432\,864(40, 28, 9, 2, 1)$
$+35\,904(40, 28, 10, 0, 2) - 11\,112(40, 28, 10, 1, 0, 1)$
$+94\,048(40, 29, 6, 5) + 197\,184(40, 29, 7, 3, 1)$
$+18\,960(40, 29, 8, 1, 2) + 2736(40, 29, 8, 2, 0, 1)$
$+9120(40, 30, 4, 6) + 71\,328(40, 30, 5, 4, 1)$
$+20\,496(40, 30, 6, 2, 2) - 19\,952(40, 30, 6, 3, 0, 1)$
$+5472(40, 31, 3, 5, 1) + 2928(40, 31, 4, 3, 2)$
$-688(40, 32, 0, 8) + 96(40, 32, 1, 6, 1)$
$-3600(40, 32, 2, 4, 2) + 1248(41, 26, 15)$
$+161\,856(41, 27, 13, 1) + 253\,536(41, 28, 11, 2)$
$+200\,448(41, 28, 12, 0, 1) + 1\,190\,592(41, 29, 9, 3)$
$+1\,123\,104(41, 29, 10, 1, 1) - 91\,920(41, 29, 11, 0, 0, 1)$
$+763\,776(41, 30, 7, 4) + 632\,352(41, 30, 8, 2, 1)$
$-9216(41, 30, 9, 0, 2) - 100\,704(41, 30, 9, 1, 0, 1)$
$+102\,336(41, 31, 5, 5) + 218\,880(41, 31, 6, 3, 1)$
$+2976(41, 31, 7, 1, 2) + 8352(41, 32, 3, 6)$
$+14\,688(41, 32, 4, 4, 1) - 49\,632(41, 32, 5, 2, 2)$
$-9888(41, 32, 5, 3, 0, 1) + 864(41, 33, 2, 5, 1)$
$+672(41, 34, 1, 4, 2) + 145\,920(42, 28, 14)$
$+572\,624(42, 29, 12, 1) + 4\,335\,504(42, 30, 10, 2)$

$+1\,079\,616(42, 30, 11, 0, 1)+4\,324\,560(42, 31, 8, 3)$
$+948\,448(42, 31, 9, 1, 1)-165\,296(42, 31, 10, 0, 0, 1)$
$+530\,448(42, 32, 6, 4)-455\,424(42, 32, 7, 2, 1)$
$-89\,112(42, 32, 8, 0, 2)-41\,472(42, 32, 8, 1, 0, 1)$
$+60\,288(42, 33, 4, 5)-202\,080(42, 33, 5, 3, 1)$
$+96(42, 33, 6, 1, 2)+2112(42, 34, 2, 6)$
$-92\,256(42, 34, 3, 4, 1)-41\,436(42, 34, 4, 2, 2)$
$-2592(42, 36, 0, 4, 2)+527\,040(43, 30, 13)$
$+9\,185\,856(43, 31, 11, 1)+15\,881\,216(43, 32, 9, 2)$
$-130\,944(43, 32, 10, 0, 1)+427\,992(43, 33, 7, 3)$
$-4\,453\,248(43, 33, 8, 1, 1)+121\,856(43, 33, 9, 0, 0, 1)$
$-302\,112(43, 34, 5, 4)-1\,135\,584(43, 34, 6, 2, 1)$
$-26\,544(43, 34, 7, 0, 2)-119\,520(43, 35, 3, 5)$
$-750\,272(43, 35, 4, 3, 1)-43\,296(43, 36, 2, 4, 1)$
$+9\,526\,708(44, 32, 12)+35\,072\,904(44, 33, 10, 1)$
$-13\,370\,376(44, 34, 8, 2)-8\,463\,808(44, 34, 9, 0, 1)$
$-12\,452\,672(44, 35, 6, 3)-5\,266\,920(44, 35, 7, 1, 1)$
$+147\,912(44, 35, 8, 0, 0, 1)-2\,199\,108(44, 36, 4, 4)$
$-377\,904(44, 36, 5, 2, 1)-1392(44, 36, 6, 0, 2)$
$-129\,288(44, 37, 2, 5)+74\,216(44, 37, 3, 3, 1)$
$+39\,953\,856(45, 34, 11)-71\,196\,440(45, 35, 9, 1)$
$-87\,400\,344(45, 36, 7, 2)-4\,588\,608(45, 36, 8, 0, 1)$
$-8\,306\,616(45, 37, 5, 3)+3\,483\,936(45, 37, 6, 1, 1)$
$-2\,171\,336(45, 38, 3, 4)-20\,496(45, 38, 4, 2, 1)$
$+645\,072(45, 39, 2, 3, 1)-136\,593\,576(46, 36, 10)$
$-324\,102\,912(46, 37, 8, 1)-12\,757\,616(46, 38, 6, 2)$
$+19\,383\,584(46, 38, 7, 0, 1)+6\,526\,728(46, 39, 4, 3)$
$+4\,042\,992(46, 39, 5, 1, 1)+3\,606\,652(46, 40, 2, 4)$
$-575\,911\,048(47, 38, 9)+163\,003\,968(47, 39, 7, 1)$
$+130\,905\,984(47, 40, 5, 2)+8\,606\,248(47, 40, 6, 0, 1)$
$+5\,025\,864(47, 41, 3, 3)+376\,800(47, 41, 4, 1, 1)$
$+1\,905\,736(47, 42, 1, 4)+741\,626\,172(48, 40, 8)$
$+925\,465\,672(48, 41, 6, 1)+45\,631\,044(48, 42, 4, 2)$

$-13\,850\,952(48, 42, 5, 0, 1)+622\,656(48, 43, 2, 3)$

$-2\,514\,974(48, 44, 0, 4)+2\,797\,322\,496(49, 42, 7)$

$-150\,379\,336(49, 43, 5, 1)-55\,372\,800(49, 44, 3, 2)$

$-2\,113\,512(49, 44, 4, 0, 1)-2\,257\,189\,656(50, 44, 6)$

$-1\,015\,905\,736(50, 45, 4, 1)-19\,598\,016(50, 46, 2, 2)$

$-6\,124\,195\,272(51, 46, 5)+59\,225\,128(51, 47, 3, 1)$

$+4\,109\,773\,668(52, 48, 4)+359\,857\,176(52, 49, 2, 1)$

$+6\,146\,784\,912(53, 50, 3)-3\,985\,650\,804(54, 52, 2)$

$-2\,281\,561\,656(55, 54, 1)+1\,518\,506\,614\tfrac{1}{7}(56, 56)$

$L_1=u^4$

$L_2=4u^7-4\tfrac{1}{2}u^8$

$L_3=28u^{10}-64u^{11}+36\tfrac{1}{3}u^{12}$

$L_4=6u^{12}+228u^{13}-834u^{14}+972u^{15}-372\tfrac{1}{4}u^{16}$

$L_5=144u^{15}+1850u^{16}-10\,464u^{17}+19\,140u^{18}-15\,024u^{19}+4354\tfrac{1}{5}u^{20}$

$L_6=60u^{17}+2320u^{18}+13\,036u^{19}-125\,398u^{20}+328\,697\tfrac{1}{3}u^{21}$
$-400\,704u^{22}+237\,408u^{23}-55\,419\tfrac{1}{2}u^{24}$

$L_7=32u^{19}+2004u^{20}+30\,000u^{21}+54\,432u^{22}-1\,410\,432u^{23}$
$+5\,155\,938u^{24}-8\,789\,456u^{25}+8\,034\,336u^{26}-3\,824\,832u^{27}$
$+747\,978\tfrac{1}{7}u^{28}$

$L_8=4u^{21}+1956u^{22}+40\,276u^{23}+319\,563u^{24}-455\,892u^{25}$
$-14\,467\,542u^{26}+75\,016\,812u^{27}-170\,379\,633u^{28}+215\,181\,212u^{29}$
$-157\,334\,718u^{30}+62\,618\,200u^{31}-10\,540\,238\tfrac{1}{8}u^{32}$

$L_9=160u^{23}+1062u^{24}+57\,712u^{25}+608\,528u^{26}+2\,619\,336u^{27}$
$-16\,460\,423u^{28}-127\,484\,176u^{29}+1\,015\,050\,965\tfrac{1}{3}u^{30}$
$-3\,011\,717\,552u^{31}+4\,970\,668\,372u^{32}-4\,985\,714\,581\tfrac{1}{3}u^{33}$
$+3\,037\,669\,968u^{34}-1\,038\,824\,088u^{35}+153\,524\,717\tfrac{1}{9}u^{36}$

$L_{10}=192u^{25}+4528u^{26}+55\,656u^{27}+1\,177\,646u^{28}+7\,246\,288u^{29}$
$+10\,576\,080u^{30}-287\,545\,744u^{31}-798\,749\,199u^{32}$
$+12\,690\,123\,948u^{33}-49\,277\,316\,720u^{34}+103\,137\,629\,280\tfrac{4}{5}u^{35}$
$-133\,782\,257\,904u^{36}+111\,287\,175\,076u^{37}-58\,117\,469\,344u^{38}$
$+17\,424\,845\,160u^{39}-2\,295\,494\,943\tfrac{9}{10}u^{40}$

$L_{11}=288u^{27}+9280u^{28}+125\,712u^{29}+1\,581\,216u^{30}+18\,714\,448u^{31}$
$+63\,812\,522u^{32}-152\,462\,256u^{33}-3\,878\,236\,540u^{34}$

$$+627\,107\,608u^{35}+143\,879\,065\,161u^{36}-750\,968\,633\,968u^{37}$$
$$+1\,964\,647\,691\,856u^{38}-3\,184\,342\,382\,352u^{39}$$
$$+3\,403\,357\,452\,512u^{40}-2\,418\,447\,158\,672u^{41}$$
$$+1\,105\,133\,298\,216u^{42}-294\,998\,985\,816u^{43}$$
$$+35\,059\,000\,785\tfrac{1}{11}u^{44}$$

$$L_{12}=36u^{28}+256u^{29}+19\,936u^{30}+283\,612u^{31}+3\,161\,046u^{32}$$
$$+32\,009\,824u^{33}+240\,321\,602u^{34}+240\,182\,836u^{35}$$
$$-4\,859\,812\,636u^{36}-42\,778\,176\,072u^{37}+133\,966\,541\,826u^{38}$$
$$+1\,414\,303\,855\,428u^{39}-10\,658\,832\,462\,547u^{40}$$
$$+34\,779\,805\,461\,120u^{41}-69\,017\,960\,261\,042u^{42}$$
$$+91\,375\,253\,899\,420u^{43}-83\,057\,128\,529\,566u^{44}$$
$$+51\,513\,943\,055\,764u^{45}-20\,925\,070\,222\,888u^{46}$$
$$+5\,033\,793\,720\,564u^{47}-544\,953\,048\,519\tfrac{1}{12}u^{48}$$

$$L_{13}=96u^{30}+2080u^{31}+28\,576u^{32}+728\,864u^{33}+7\,005\,320u^{34}$$
$$+66\,578\,984u^{35}+507\,532\,948u^{36}+2\,423\,286\,328u^{37}$$
$$-5\,404\,219\,620u^{38}-82\,510\,922\,288u^{39}-363\,012\,886\,140u^{40}$$
$$+2\,961\,836\,513\,480u^{41}+10\,521\,866\,653\,872u^{42}$$
$$-139\,956\,115\,158\,584u^{43}+575\,861\,898\,452\,491u^{44}$$
$$-1\,383\,898\,026\,988\,504u^{45}+2\,217\,588\,585\,252\,548u^{46}$$
$$-2\,482\,806\,223\,917\,024u^{47}+1\,963\,454\,828\,332\,664u^{48}$$
$$-1\,080\,387\,949\,344\,328u^{49}+394\,989\,766\,870\,668u^{50}$$
$$-86\,479\,507\,957\,568u^{51}+8\,596\,964\,155\,137\tfrac{1}{13}u^{52}$$

$$L_{14}=32u^{31}+88u^{32}+7464u^{33}+91\,552u^{34}+1\,371\,504u^{35}+19\,384\,396u^{36}$$
$$+146\,503\,788u^{37}+1\,163\,233\,672u^{38}+6\,464\,293\,172u^{39}$$
$$+15\,554\,510\,188u^{40}-169\,391\,947\,684u^{41}-1\,047\,905\,712\,244u^{42}$$
$$-1\,526\,151\,248\,604u^{43}+47\,026\,432\,198\,358u^{44}$$
$$+19\,421\,489\,941\,908u^{45}-1\,671\,228\,453\,101\,636u^{46}$$
$$+8\,935\,920\,861\,996\,628u^{47}-25\,923\,646\,975\,358\,927u^{48}$$
$$+49\,545\,267\,625\,587\,224\tfrac{4}{7}u^{49}-66\,656\,733\,000\,047\,148u^{50}$$
$$+64\,670\,692\,272\,214\,948u^{51}-45\,263\,326\,771\,350\,522u^{52}$$
$$+22\,381\,377\,920\,023\,624u^{53}-7\,439\,170\,129\,307\,332u^{54}$$
$$+1\,494\,464\,464\,035\,644u^{55}-137\,345\,637\,320\,093\tfrac{9}{14}u^{56}$$

$$
\begin{aligned}
L_{15} = {} & 16u^{32}+1792u^{34}+13\,664u^{35}+327\,184u^{36}+3\,461\,344u^{37} \\
& +43\,230\,912u^{38}+417\,871\,152u^{39}+2\,606\,217\,284u^{40} \\
& +16\,981\,144\,216u^{41}+63\,096\,311\,348u^{42}-50\,438\,332\,344u^{43} \\
& -3\,077\,640\,717\,280u^{44}-9\,805\,663\,359\,448u^{45} \\
& +22\,286\,425\,833\,636u^{46}+613\,239\,401\,603\,216u^{47} \\
& -1\,292\,278\,599\,099\,041\tfrac{1}{3}u^{48}-17\,435\,540\,294\,420\,040u^{49} \\
& +129\,670\,417\,469\,643\,861\tfrac{3}{5}u^{50}-456\,290\,659\,834\,772\,664u^{51} \\
& +1\,031\,293\,051\,074\,360\,323u^{52}-1\,639\,758\,478\,157\,185\,960u^{53} \\
& +1\,902\,359\,776\,265\,120\,460u^{54}-1\,628\,702\,401\,831\,872\,924\tfrac{4}{5}u^{55} \\
& +1\,022\,447\,022\,976\,122\,908u^{56}-459\,043\,869\,152\,433\,584u^{57} \\
& +139\,871\,408\,486\,324\,800u^{58}-25\,959\,418\,944\,366\,176u^{59} \\
& +2\,218\,295\,308\,971\,345\tfrac{3}{5}u^{60}.
\end{aligned}
$$

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